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Necessary and sufficient condition for additivity in the sense of the Palmgren–Miner rule

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Abstract

Formal conditions are established regarding the compatibility of the Palmgren–Miner rule with laws describing the damage development during fatigue. The Palmgren–Miner rule can be used for fatigue life predictions, if and only if, the damage development rate can be presented as a product of the functions of stress (strain) amplitude and current amount of damage. Important properties of the additive damage development laws are derived and methods are proposed for checking whether a particular damage development law is consistent with the Palmgren–Miner rule. Some of the reasons for inaccurate fatigue life predictions based on the Palmgren–Miner rule are rationalised using the derived condition for additivity. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The life expectancies of parts subjected to spectrum loading are commonly estimated [1–6] from the empirical Palmgren–Miner damage law, introduced first by Palmgren [1] in the analysis of ball bearings and adapted by Miner [2] for aircraft structures.

Let n_i denote the number of stress cycles with a constant amplitude $\Delta\sigma_i$ and n_i^f is the number of cycles with the same amplitude to a specified extent of the damage ξ after which the component is considered to have failed. In case of fatigue controlled by crack propagation, the nature of the damage ξ can be a fatigue crack of specified length which can be considered as a failure, for example in an aircraft structural component [2]. Miner [2] suggested that in a fatigue test at a constant stress amplitude $\Delta\sigma_i$ damage could be regarded to accumulate linearly with the number of cycles. Accordingly, if at a stress amplitude $\Delta\sigma_1$ the component has a life of n_1^f cycles, which corresponds to the amount of damage ξ, after n_1 cycles at the same stress amplitude, the amount of damage will be $(n_1/n_1^f)\xi$. After n_2 stress cycles spent at a stress amplitude $\Delta\sigma_2$, characterised by a fatigue life of n_2^f cycles, the amount of damage will be $(n_2/n_2^f)\xi$, etc. [3]. Failure occurs when, at a certain stress amplitude $\Delta\sigma_M$, the sum of the partial amounts of damage attains the amount ξ, i.e., when

$$\frac{n_1}{n_1^f}\xi + \frac{n_2}{n_2^f}\xi + \cdots + \frac{n_M}{n_M^f}\xi = \xi \qquad (1)$$

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is fulfilled. As a result, the analytical expression of the Palmgren–Miner rule becomes

$$\sum_{i=1}^{M} n_i/n_i^f = 1, \tag{2}$$

where n_i^f is the number of cycles needed to attain the specified amount of damage ξ at a constant stress amplitude $\Delta\sigma_i$. Essentially, Eq. (2) is an additivity rule according to which the total number of stress cycles required to attain a specified level of damage ξ is obtained by adding the absolute durations n_i spent at each stress amplitude $\Delta\sigma_i$, until the sum of the relative durations n_i/n_i^f becomes unity. The fraction of expended fatigue life at a particular stress amplitude is the ratio of the number of stress cycles spent at this stress amplitude and the total number of stress cycles of the same stress amplitude needed to attain the specified level of damage (from zero initial damage). It is also assumed that the sequence in which the various stress amplitudes are imposed does not affect the fatigue life.

Following the observation that the fatigue life depends on the sequence in which the loading blocks are applied [6] and experimental evidence from cases where the Palmgren–Miner rule fails to make correct life predictions, the correctness of this rule is often questioned [7]. In order to improve the predictions based on the Palmgren–Miner rule, double-linear damage rules were introduced, initially associated with the crack initiation and crack propagation stage of the fatigue crack development [8]. Alternatively, a double exponential law was introduced for the accumulation of damage during each of the phases crack initiation and stage I propagation [9].

Discrepancies and difficulties associated with determining the kneepoint between the two stages, however, prompted the development of double-linear damage rules related to two subsequent phases of the fatigue crack development, not necessarily coinciding with the 'crack initiation' and 'crack propagation' stage [10].

Deviations from the Palmgren–Miner rule have also been reported during two-stage cumulative damage tests due to changing the loading conditions from push-pull to torsion [11]. Furthermore, results were reported showing that on changing the stress levels from high to low, due to development of crack tip residual stresses, the theoretical predictions may become conservative [10].

Some of the causes for these discrepancies will be discussed later. It will be demonstrated that a necessary condition for the use of the Palmgren–Miner rule for predicting the fatigue life is a particular factorisation of the damage development law.

Suppose that the damage development rate can be presented in the general form:

$$\mathrm{d}\xi(n)/\mathrm{d}n = \varphi(\xi, p), \tag{3}$$

where $\varphi(\xi, p)$ is a non-negative function, n is the time or cycle, $\xi(n)$ is the amount of damage after n cycles and p is a parameter which varies with cycles (time) – usually the stress (or strain) amplitude. The damage development law given by Eq. (3) will be referred to as 'autonomous' because it does not explicitly involve the independent variable n.

In cases where the fatigue life is dominated by a crack propagation (e.g. high stressed cast aluminium alloys [12] or other alloys containing crack-like defects), the fatigue damage consists solely of fatigue crack growth. The fatigue crack initiation life is negligible, the damage will be nothing but crack growth and its amount ξ will be equal to the crack size a ($\xi \equiv a$).

A widely used fatigue crack growth model is the Paris–Erdogan [13] power law:

$$\mathrm{d}a(n)/\mathrm{d}n = C\Delta K^m, \tag{4}$$

where $\Delta K = Y\Delta\sigma\sqrt{\pi a}$ is the stress intensity factor range, C and m are material constants and Y is a parameter which depends on a.

If crack closure is present, the relationship

$$\mathrm{d}a(n)/\mathrm{d}n = C(\Delta K_{\mathrm{eff}})^m (K_{\max})^p \tag{5}$$

can be applied [14], where $\Delta K_{\mathrm{eff}} = K_{\max} - K_{\mathrm{op}}$ is the effective stress intensity factor range, $K_{\max}$ is the maximum stress intensity factor and K_{op} is the stress intensity factor required to open the fatigue crack. Eqs. (4) and (5) are special cases of the general Eq. (3), in which the parameter p is the stress range $\Delta\sigma$.

In order to characterise the development of physically small cracks, the equation

$$\mathrm{d}a(n)/\mathrm{d}n = B\Delta\gamma a^{\beta} - D \quad (6)$$

has been discussed [15], where B and β are material constants, D represents threshold and $\Delta\gamma$ is the applied shear strain range. Eq. (6) is also a special case of the general Eq. (3) in which the parameter p now stands for the strain range $\Delta\gamma$.

Ibrahim and Miller [16] showed that the elastic–plastic fracture mechanics relationship

$$\mathrm{d}a(n)/\mathrm{d}n = 3.2(\Delta\gamma_{\mathrm{p}})^2 a, \quad (7)$$

(where $\Delta\gamma_{\mathrm{p}}$ is the plastic shear strain range) can be used to describe the damage accumulation in the short crack propagation phase for 0.4% C steel in the hot-rolled condition. It has been reported [11] that in a medium carbon steel, the growth rate of short cracks of length a can be expressed by

$$\mathrm{d}a(n)/\mathrm{d}n = C_1(d_i - a) \quad (8)$$

valid for microstructurally short cracks and by

$$\mathrm{d}a(n)/\mathrm{d}n = C_2 a - D \quad (9)$$

valid for the physically small cracks, where C_1 and C_2 are material constants depending on the stress level, d_i is a microstructural parameter and D is a threshold condition.

If the damage development function $\varphi(\xi, p)$ in Eq. (3) is known, the amount of damage ξ can be determined by analytical or numerical integration. In cases where the $\varphi(\xi, p)$ is unknown or its integration is too complicated, the Palmgren–Miner rule (2) is often used due to its simplicity. Despite the wide use of the Palmgren–Miner rule (2), no theoretical analysis exists regarding its compatibility with equations describing damage development. The aim of this work is establishing formal conditions under which the Palmgren–Miner rule can be applied to predict the fatigue life, instead of direct integration of the damage curves, or stated alternatively – establishing conditions for additivity of the damage development laws in the sense of the Palmgren–Miner rule.

2. Necessary and sufficient condition for additivity in the sense of the Palmgren–Miner rule

It will be established that the Palmgren–Miner rule is compatible with the damage development law (3), if and only if, the damage development rate $\mathrm{d}\xi(n)/\mathrm{d}n$ at a constant parameter p (stress $\Delta\sigma$ or strain $\Delta\gamma$ amplitude) can be presented (factorised) as a product of a function $\varphi_1(\xi)$ of the current amount of damage and a function $\varphi_2(p)$ of the parameter (amplitude) p:

$$\frac{\mathrm{d}\xi(n)}{\mathrm{d}n} = \varphi_1(\xi)\varphi_2(p). \quad (10)$$

A possible way to establish the validity of the above statement is by proving that if (i) (A) the damage development relationship at a constant stress amplitude can be factorised as in (10), then (B) the Palmgren–Miner rule (2) is fulfilled, i.e., proving that A is sufficient for B and (ii) if (B) the Palmgren–Miner law (2) is assumed to be true then the truth of A necessarily follows from that of B.

Assume that Eq. (10) holds and that M loading amplitudes $p_1 = \Delta\sigma_1, p_2 = \Delta\sigma_2, \ldots, p_M = \Delta\sigma_M$ are applied sequentially. Under the first amplitude the damage increases from 0 to ξ_1, under the second amplitude – from ξ_1 to ξ_2, etc., until the final amount of damage ξ_M is attained. The number of cycles corresponding to each stress or strain amplitude needed to increase the damage to amounts $\xi_1, \xi_2, \ldots, \xi_M$ are $n_1, n_2, \ldots, n_M$, respectively. These can be obtained from the direct integration of Eq. (10). If the latter equation is rearranged as $\mathrm{d}\xi/\varphi_1(\xi) = \varphi_2(p)\,\mathrm{d}n$, integrating both sides at $p = \text{const.}$, within limits $0, \xi$ and $0, n$ yields

$$n = \frac{1}{\varphi_2(p)} \int_0^{\xi} \mathrm{d}\xi/\varphi_1(\xi). \quad (11)$$

If $F(\xi) = \int_0^{\xi} \mathrm{d}\xi/\varphi_1(\xi)$, then

$$n_i = \frac{1}{\varphi_2(p_i)} \int_{\xi_{i-1}}^{\xi_i} \mathrm{d}\xi/\varphi_1(\xi) = \frac{1}{\varphi_2(p_i)}[F(\xi_i) - F(\xi_{i-1})], \quad i = 1, M, \quad (12)$$

where the parameter p_i in the ith interval is the corresponding stress amplitude $\Delta\sigma_i$ ($p_i \equiv \Delta\sigma_i$). The number of cycles N_i at the stress amplitude

$p_i = \Delta\sigma_i$ needed to extend the amount of damage from zero ($\xi_0 = 0$) to ξ_M is

$$N_i = \frac{1}{\varphi_2(p_i)} \int_0^{\xi_M} \mathrm{d}\xi/\varphi_1(\xi) = \frac{1}{\varphi_2(p_i)} F(\xi_M), \quad i = 1, M. \tag{13}$$

As can be verified directly

$$\sum_{i=1}^{M} n_i/N_i = \frac{1}{F(\xi_M)} \sum_{i=1}^{M} [F(\xi_i) - F(\xi_{i-1})] = \frac{F(\xi_M)}{F(\xi_M)} = 1, \tag{14}$$

where n_i and N_i are given by Eqs. (12) and (13), respectively. Eq. (14) holds irrespective of the number of loading blocks or the number of cycles n_i into them. Hence, condition (10) is sufficient for additivity in the sense of the Palmgren–Miner rule.

While establishing that condition (10) is also a necessary condition it is assumed that if the Palmgren–Miner equality (2) holds, it holds for any number of loading blocks containing any number of cycles. Consequently, if Eq. (2) holds, it will also hold for the simplest spectral loading which combines only two stress (strain) amplitude blocks (Fig. 1(a)). At the first amplitude p_1, the damage development is given by the continuous and monotonic function $\xi(n, p_1)$. After n_1 cycles, corresponding to an amount of damage ξ_0, the parameter (amplitude) is changed to p_2, for which the amount of damage is described by $\xi(n, p_2)$. The course of damage development at the amplitude p_2 is assumed to be the same as if the amount of damage ξ_0 had all been produced at a constant amplitude p_2. The number of stress cycles necessary to attain a damage increment $\Delta\xi$ regarding ξ_0 ($\xi_f = \xi_0 + \Delta\xi$) during fatigue cycling at amplitudes p_1 and p_2 are the increments in the number of cycles (time) Δn_1 and Δn_2 ($n_{f1} = n_1 + \Delta n_1$; $n_{f2} = n_2 + \Delta n_2$), respectively (Fig. 1(a)). According to the initial assumption, Eq. (2) is valid for any number of loading blocks containing any number of cycles, hence

$$\frac{n_1}{n_1 + \Delta n_1} + \frac{\Delta n_2}{n_2 + \Delta n_2} = 1 \tag{15}$$

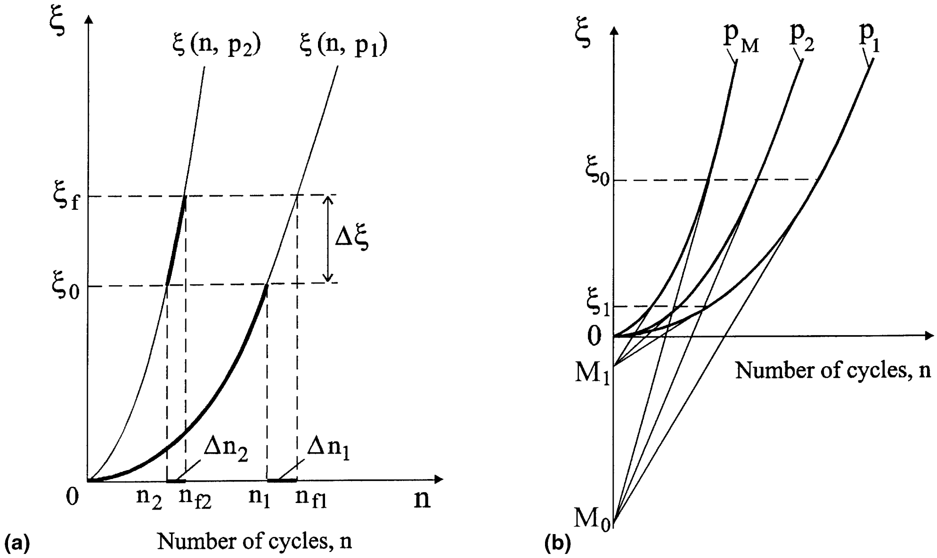

Fig. 1. (a) Fatigue damage development for the simplest spectral loading which combines only two stress (strain) amplitude blocks. At the first amplitude p_1, the damage is given by the continuous and monotonic function $\xi(n, p_1)$; at the second amplitude p_2 (which starts after n_1 cycles, corresponding to amount of damage ξ_0) the amount of damage is given by $\xi(n, p_2)$. (b) Geometrical condition for consistency of the damage development law with the Palmgren–Miner rule. The tangents to points corresponding to the same amount of damage ξ_0 and ξ_1 intersect at common points M_0 and M_1.

will be valid, which transforms into

$$\frac{\Delta n_1}{\Delta n_2} = \frac{n_1}{n_2}. \qquad (16)$$

Let $n = \varphi(\xi, p)$ be the inverse of the monotonic function $\xi = \xi(n, p)$ describing the damage development as a function of the number of cycles n at a constant amplitude p. Since

$$\Delta n_1 = n_{f1} - n_1 = \varphi(\xi_f, p_1) - \varphi(\xi_0, p_1)$$

and

$$\Delta n_2 = n_{f2} - n_2 = \varphi(\xi_f, p_2) - \varphi(\xi_0, p_2)$$

(Fig. 1(a)), Eq. (16) can also be presented as

$$\frac{\varphi(\xi_f, p_1)}{\varphi(\xi_0, p_1)} = \frac{\varphi(\xi_f, p_2)}{\varphi(\xi_0, p_2)} \qquad (17)$$

valid for any chosen ξ_0 between zero and ξ_f and for any two parameters p_1 and p_2. This is only possible if $\varphi(\xi, p)$ can be factorised as

$$n \equiv \varphi(\xi, p) = g(\xi)h(p). \qquad (18)$$

Differentiating Eq. (18) with respect to n results in $\mathrm{d}\xi/\mathrm{d}n = [1/h(p)][1/(\mathrm{d}g(\xi)/\mathrm{d}\xi)]$, which is exactly Eq. (10) in which $\varphi_1(\xi) \equiv 1/g'(\xi)$ and $\varphi_2(p) \equiv 1/h(p)$.

Thus, from the assumption that the Palmgren–Miner rule holds for any number of loading blocks it follows that the Palmgren–Miner rule also holds for two loading blocks, from which Eqs. (18) and (10) follow. Hence, Eq. (10) necessarily follows from the assumption that the Palmgren–Miner rule holds. This means that if the factorisation in Eq. (10) is impossible, there will exist a division of the loading blocks (a division into two loading blocks, for example) for which the Palmgren–Miner rule will not hold.

Eq. (10) is obtained from Eq. (18) by differentiation. Conversely, the integration of Eq. (10) gives

$$n = \frac{1}{\varphi_2(p)} \int_0^{\xi} \mathrm{d}\xi/\varphi_1(\xi),$$

which is exactly Eq. (18) in which

$$g(\xi) \equiv \int_0^{\xi} \mathrm{d}\xi/\varphi_1(\xi)$$

and

$$h(p) \equiv \frac{1}{\varphi_2(p)}.$$

Therefore, Eqs. (18) and (10) are equivalent, hence Eq. (18) is also a necessary and sufficient condition for additivity in the sense of the Palmgren–Miner rule.

3. Properties of the additive damage development laws

If the increments in Eq. (16) are small, they can be first-order approximated as: $\Delta n_1 \approx \Delta\xi/(\mathrm{d}\xi/\mathrm{d}n)|_{n=n_1, p=p_1}$, $\Delta n_2 \approx \Delta\xi/(\mathrm{d}\xi/\mathrm{d}n)|_{n=n_2, p=p_2}$ (Fig. 1(a)), which, substituted in Eq. (16) give

$$n_1 \frac{\mathrm{d}\xi}{\mathrm{d}n}\bigg|_{n=n_1, p=p_1} = n_2 \frac{\mathrm{d}\xi}{\mathrm{d}n}\bigg|_{n=n_2, p=p_2} = f(\xi_0), \qquad (19)$$

i.e., the product $n(\mathrm{d}\xi/\mathrm{d}n)$ depends on ξ_0 and does not depend on amplitude p.

For infinitesimally small Δn_1 and Δn_2, Eq. (16) becomes

$$\frac{\mathrm{d}n_1}{n_1} = \frac{\mathrm{d}n_2}{n_2}, \qquad (20)$$

i.e., the ratio $\mathrm{d}n/n$ is only a function of ξ and does not depend on the parameter p:

$$\frac{\mathrm{d}n}{n} = F(\xi). \qquad (21)$$

In Eq. (19), $f(\xi_0)$ (the segment $\xi_0 M_0$ in Fig. 1(b)) gives the length of the segment on the coordinate axis ξ, which is limited by ξ_0 and the intersection of the tangent corresponding to ξ_0 from the damage development curve $p = p_1$, with the coordinate axis ξ. At a specified ξ, this segment is of the same length for all damage curves or, in other words, the tangents to all damage curves, at points corresponding to the same amount of damage, intersect at a common point lying on the ξ-axis (Fig. 1(b)). Taking the tangents to all damage development curves, at points corresponding to the same amount of damage, provides an immediate way to see geometrically whether a particular function describing damage development is consistent with the Palmgren–Miner rule (2). Consistency exists, if and only if, at any specified amount of damage, all the tangents intersect at a common point lying on the coordinate axis giving the amount of damage (points $M_0, M_1, \ldots$, etc., in Fig. 1(b)).

Substituting the inverse function $n = \varphi(\xi, p)$ in Eq. (21) gives

$$\frac{\varphi'(\xi,p)\,\mathrm{d}\xi}{\varphi(\xi,p)} = F(\xi) \tag{22}$$

at a constant amplitude p. Eq. (22) can also be presented as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[\ln \varphi(\xi,p)] = F(\xi),$$

which is fulfilled only if $\ln \varphi(\xi,p) = g(\xi) + h(p)$ or if $n = \varphi(\xi,p)$ can be factorised as in Eq. (10). As discussed in Appendix A, the application area of condition (10) is wide and extends beyond cumulative damage during fatigue.

Next, consider any two damage development curves corresponding to two different values of the parameter p ($\xi = \xi(n,p_1)$ and $\xi = \xi(n,p_2)$). The amount of damage corresponding to failure is ξ_f and the number of cycles needed to attain the damage ξ_f at the first (p_1) and at the second (p_2) amplitude are N_1 and N_2, respectively. Let ξ_x be any specified amount of damage between zero and ξ_f, attained after n_1 loading cycles at a constant amplitude p_1 or after n_2 cycles at a constant amplitude p_2 (Fig. 2(a)). In the previous section it was established that conditions (10) and (18) are equivalent. Therefore if $\xi = \xi(n,p)$ is to be additive in the sense of the Palmgren–Miner rule it should have the factorisation (18). Thus we can write:

$$N_1 = g(\xi_f)h(p_1), \tag{23}$$

$$n_1 = g(\xi_x)h(p_1) \tag{24}$$

for the first damage curve and

$$N_2 = g(\xi_f)h(p_2), \tag{25}$$

$$n_2 = g(\xi_x)h(p_2) \tag{26}$$

for the second curve. Dividing Eqs. (24) and (26) by Eqs. (23) and (25) results in

$$\frac{n_1}{N_1} = \frac{n_2}{N_2}, \tag{27}$$

i.e., for any two specified levels of damage, the ratio between the number of cycles necessary to attain them at the same stress (strain) amplitude is the same for any damage curve (Fig. 2(a)) ($n_1/N_1 = \cdots = n_i/N_i = \cdots = n_M/N_M$). Using Eq. (27), it can be verified easily that for a simple

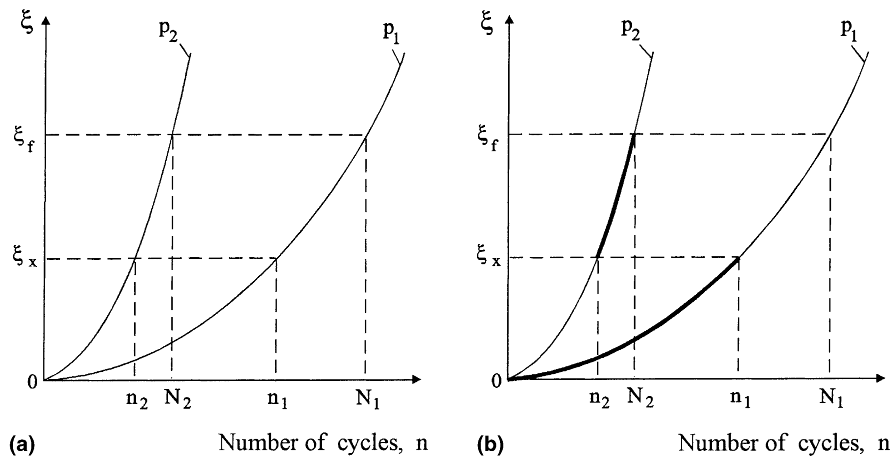

Fig. 2. (a) Another geometrical condition for consistency of the damage development law with the Palmgren–Miner rule. For any two specified levels of damage, the ratio between the number of cycles necessary to attain them at the same stress (strain) amplitude is the same for any damage curve: $n_1/N_1 = n_2/N_2$. (b) For any distribution of the number of cycles in the two loading blocks, the Palmgren–Miner summation always results in unity if the damage law has the factorisation (18).

spectral loading consisting of n_1 fatigue cycles at a constant amplitude p_1 followed by $N_2 - n_2$ cycles at a constant amplitude p_2 after attaining the amount of damage ξ_x (Fig. 2(b)), the Palmgren–Miner summation is fulfilled.

Conversely, if the damage law does not have the factorisation (18), the Palmgren–Miner summation until unity does not predict the fatigue life correctly. This is illustrated in Fig. 3 for two different Palmgren–Miner summations. In the first summation (Fig. 3(a)) $n_1 = N_1/2$ number of cycles are spent at the first (smaller) amplitude p_1 and $n_2 = N_2/2$ number of cycles are spent at the second (larger) amplitude p_2. By contrast, in the second summation (Fig. 3(b)) $n_2 = N_2/2$ number of cycles are spent first at the larger amplitude p_2, followed by $n_1 = N_1/2$ cycles at the smaller amplitude p_1.

The Palmgren–Miner summation until unity, as a way of predicting the number of cycles until failure, gives $n_1/N_1 + n_2/N_2 = 0.5 + 0.5 = 1$ and the predicted fatigue life is $n_1 + n_2 = (N_1 + N_2)/2$ cycles. As can be verified from the direct integration over the damage curves (Fig. 3(a)) the damage ξ_{lh} which corresponds to obtaining unity in the Palmgren–Miner summation is smaller than the true amount of damage at failure $\xi_f(\xi_{lh} < \xi_f)$. During low-to-high amplitude loading (Fig. 3(a)) the true number of cycles $N_1/2 + n_{2r}$ obtained from the direct integration over the damage curves is larger than the predicted $(N_1 + N_2)/2$ cycles from the Palmgren–Miner rule. On the other hand, the damage ξ_{hl} (Fig. 3(b)) from high-to-low amplitude loading (which corresponds to obtaining unity in the Palmgren–Miner summation) is larger than the true amount of damage at failure $(\xi_{hl} > \xi_f)$. During high-to-low amplitude loading (Fig. 3(b)) the true number of cycles $N_2/2 + n_{1r}$ obtained from the direct integration over the damage curves is smaller than the number of cycles $(N_1 + N_2)/2$ predicted by the Palmgren–Miner rule. Thus, in both cases the Palmgren–Miner summation predicts a wrong number of cycles to failure.

These discrepancies appear because the damage development curves do not have the factorisation (18) or (10). Indeed, if they had this factorisation, the equation

$$\frac{n_1}{N_1} = \frac{n_x}{N_2} \tag{28}$$

would hold (see Eq. (27)) where n_x is the number of cycles corresponding to an amount of damage ξ_x (Fig. 3(a)) and $n_1 = N_1/2$. Since $n_1/N_1 = 0.5$, it follows from Eq. (28) that $n_x/N_2 = 0.5$. As a result, $n_2 = n_x = N_2/2$ (because $n_2/N_2 = 0.5$ by definition)

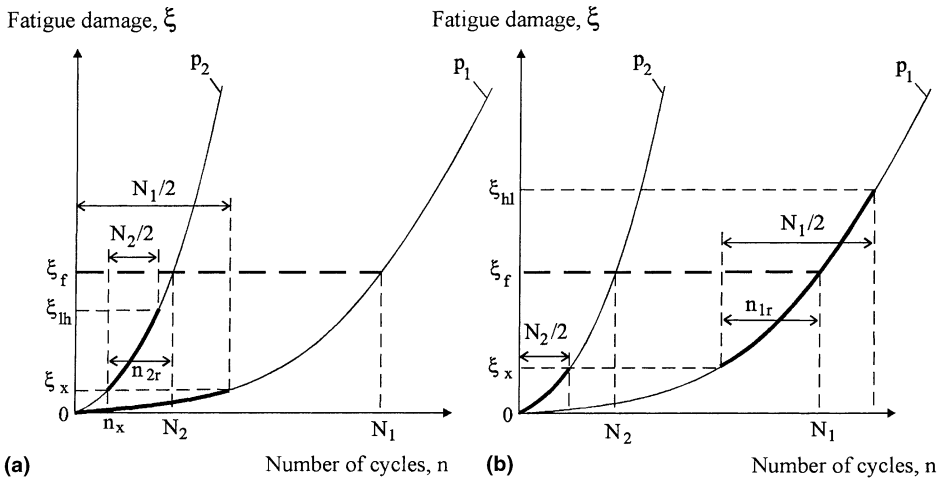

Fig. 3. If the damage law does not have the factorisation (10) or (18), the Palmgren–Miner summation until unity predicts the wrong number of cycles to failure. (a) In case of cycling from low to high amplitude the predicted number of cycles $(N_1 + N_2)/2$ falls short of the true number $N_1/2 + n_{2r}$ necessary to attain the damage at failure: $(N_1 + N_2)/2 < N_1/2 + n_{2r}$. (b) By contrast, in case of cycling from high to low amplitude the predicted number of cycles to failure $(N_1 + N_2)/2$ is greater than the true value $N_2/2 + n_{1r}$.

and the true number of cycles until damage ξ_f from the integration of the damage curves will be $n_1 + n_2 = (N_1 + N_2)/2$, which is exactly what is predicted by the Palmgren–Miner rule. Incidentally, Eq. (27) provides an alternative way to check geometrically whether a particular function describing damage development is consistent with the Palmgren–Miner rule (2). Consistency exists, if and only if, at any two specified levels of damage ξ_x and ξ_f, Eq. (27) holds for any pair of damage curves p_1 and p_2.

4. Discussion and conclusions

The necessary and sufficient condition (10) means that if such factorisation exists, the direct integration of the damage development law can be replaced by a Palmgren–Miner summation, for any division of the loading intervals and for any particular number of cycles into them. The factorisation discussed can also be regarded as a condition for compatibility of a given damage development law with the Palmgren–Miner rule. If the Palmgren–Miner rule holds for any number of loading blocks containing any number of cycles, the damage development rate should necessarily have the factorisation (10).

Conversely, if the damage development rate cannot be factorised as in (10), the damage development law is not additive in the sense of the Palmgren–Miner rule. As a consequence, there will exist a division of the loading intervals and a particular distribution of the number of cycles, such as the fatigue life predicted by the Palmgren–Miner summation will differ from that obtained from a direct integration.

Conditions (10) and (18) are equivalent because they follow from each other. Hence Eq. (18) is also a necessary and sufficient condition for additivity in the sense of the Palmgren–Miner rule. Accordingly, the latter is consistent with a particular damage development law if and only if, the cycles (time) in the damage development law can be presented as a product of a function of the amount of damage and a function of the stress amplitude.

The derivations have been based on the general damage development law (3) which is also applicable to cases where most of the fatigue life is spent on crack initiation or on development of physically small cracks (see Eqs. (6) and (9)).

Consider fatigue damage consisting solely of fatigue crack growth described by the Paris–Erdogan Eq. (4). Since the power m in Eq. (4) does not depend on the crack size a, the fatigue crack growth rate can be factorised as in (10) and therefore the Paris–Erdogan law is additive in the sense of the Palmgren–Miner rule which has also been confirmed experimentally. Frost et al. [3] demonstrated that for two loading blocks, the Palmgren–Miner law is derivable from considerations of simple fatigue crack propagation described by the Paris–Erdogan power law (4). In order to verify this in the general case, let us assume M loading amplitudes $\Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_M$ applied sequentially. Suppose, for the sake of simplicity that the parameter Y is constant. During fatigue loading at the first stress amplitude, a crack with an initial size a_0 extends until size a_1; during loading at the second stress amplitude the crack extends from size a_1 until size a_2, etc., until the final crack size a_M is attained. The number of cycles corresponding to each stress amplitude needed to extend the fatigue crack until sizes $a_1, a_2, \ldots, a_M$ is $n_1, n_2, \ldots, n_M$, respectively. These can be obtained by a direct integration of the Paris–Erdogan law (4). Thus, for $m \neq 2$

$$n_i = \int_{a_{i-1}}^{a_i} \frac{\mathrm{d}a}{C(Y\Delta\sigma_i\sqrt{\pi a})^m} = \frac{K}{\Delta\sigma_i^m}\left[\frac{1}{a_{i-1}^r} - \frac{1}{a_i^r}\right], \quad i = 1, M, \tag{29}$$

where K is a constant, $r = (m-2)/2$, and $i = 1, M$. The number of cycles N_i at the stress amplitude $\Delta\sigma_i$ needed to extend the crack from the initial size a_0 to the final size a_M is

$$N_i = \int_{a_0}^{a_M} \frac{\mathrm{d}a}{C(Y\Delta\sigma_i\sqrt{\pi a})^m} = \frac{K}{\Delta\sigma_i^m}\left[\frac{1}{a_0^r} - \frac{1}{a_M^r}\right], \tag{30}$$

where $r = (m-2)/2$ and $i = 1, M$. As can be verified directly

$$\sum_{i=1}^{M} n_i/N_i = \frac{1}{1/a_0^r - 1/a_M^r}\sum_{i=1}^{M}\left[\frac{1}{a_{i-1}^r} - \frac{1}{a_i^r}\right] = 1, \tag{31}$$

where n_i and N_i are given by Eqs. (29) and (30), respectively. In a similar way, the validity of Eq. (31) is established for the case $m = 2$.

It needs to be pointed out that conditions (10) or (18) are only conditions for compatibility of the Palmgren–Miner rule with a particular damage development law. If the damage development law does not predict correctly the fatigue life, the Palmgren–Miner rule, although compatible with it, will also fail to make correct fatigue life predictions. However, if the Palmgren–Miner rule fails to make correct prediction of the fatigue life, this may well be due to the fact that the true damage development rate does not have the factorisation (10) for some part or for the whole fatigue process. Eq. (6) for example, describing the development of physically small cracks, is not additive in the sense of the Palmgren–Miner rule because it cannot be factorised as in Eq. (10). On the other hand, Eq. (7) giving the damage accumulation in the short crack propagation phase can be factorised and is compatible with the Palmgren–Miner rule. Even for the same steel some of the relevant damage development laws can be factorised as in Eq. (10) (see Eq. (8)) whereas others cannot (see Eq. (9)). Thus, Eq. (9) is not compatible with the Palmgren–Miner rule and the latter should not be applied to describe damage development during growth of physically small cracks in a medium carbon steel.

It can be verified easily that the Palmgren–Miner rule is still valid even if the damage development rate depends on more than one parameter, as long as the factorisation

$$\mathrm{d}\xi(n)/\mathrm{d}n = \varphi_1(\xi)\varphi_2(p_1, p_2, \ldots, p_s) \tag{32}$$

is possible, where $p_1, p_2, \ldots, p_s$ are parameters which depend only on the current state of the fatigue loading but do not depend on loading history and time.

If Eq. (32) is rearranged as $\mathrm{d}\xi/\varphi_1(\xi) = \varphi_2(p_1, p_2, \ldots, p_s)\,\mathrm{d}n$, the integration of both sides at $p_i = \text{const.}$, $i = 1, s$, within limits $0, \xi$ and $0, n$, yields

$$n = \frac{1}{\varphi_2(p_1, \ldots, p_s)} \int_0^{\xi} \mathrm{d}\xi/\varphi_1(\xi). \tag{33}$$

If $F(\xi) = \int_0^{\xi} \mathrm{d}\xi/\varphi_1(\xi)$, for the kth interval of crack growth characterised by constant loading parameters $p_i^{(k)} = \text{const.}$, $i = 1, s$,

$$\begin{aligned} n_k &= \frac{1}{\varphi_2(p_1^{(k)}, \ldots, p_s^{(k)})} \int_{\xi_{k-1}}^{\xi_k} \mathrm{d}\xi/\varphi_1(\xi) \\ &= \frac{1}{\varphi_2(p_1^{(k)}, \ldots, p_s^{(k)})} [F(\xi_k) - F(\xi_{k-1})], \quad k = 1, M. \end{aligned} \tag{34}$$

Since the number of cycles N_k needed to extend the amount of damage from zero ($\xi_0 = 0$) to ξ_M is

$$\begin{aligned} N_k &= \frac{1}{\varphi_2(p_1^{(k)}, \ldots, p_s^{(k)})} \int_0^{\xi_M} \mathrm{d}\xi/\varphi_1(\xi) \\ &= \frac{1}{\varphi_2(p_1^{(k)}, \ldots, p_s^{(k)})} F(\xi_M), \quad k = 1, M \end{aligned} \tag{35}$$

and is characterised by the same factor $1/\varphi_2(p_1^{(k)}, \ldots, p_s^{(k)})$ as n_k in Eq. (34). As can be verified directly, Eq. (14) is fulfilled where index i is substituted by index k.

The autonomous damage development law (3) expresses the fact that the rate of the damage development $\xi_n(n)$ under variable parameter p, at the instant corresponding to amount of damage ξ, can be approximated by the instantaneous rate of damage development at constant p:

$$\mathrm{d}\xi_n(n)/\mathrm{d}n = \mathrm{d}\varphi(\xi, p)/\mathrm{d}n_{n=n(\xi), p=p(n)}.$$

In this sense, the autonomous damage development law can be regarded as 'additive' in a general sense. Additivity in a general sense implies that damage development under variable parameter (amplitude) can be approximated stepwise by damage development laws characterised by a constant parameter (amplitude) at each approximation step. Consequently, the damage laws can be classified into the wide class of damage laws additive in a general sense and the subset of damage laws additive also in a sense of Palmgren–Miner. Laws additive in a sense of Palmgren–Miner are additive in a general sense, whereas additivity in a general sense does not necessarily mean additivity in a sense of Palmgren–Miner.

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Appendix A

The derivations are also valid for any continuous process depending on a parameter p. Suppose the product ξ from the process, at a constant parameter p, is given by the continuous and monotonous function $\xi(\tau, p)$, where τ denotes time.

If the parameter p varies with time ($p = p(\tau)$), in order to find the time τ_f necessary to attain a particular amount ξ_f of the product, the additivity rule

$$\sum \frac{\Delta\tau}{\tau_{\xi_f}(p)} = 1 \qquad (A.1)$$

can be applied, where $\tau_{\xi_f}(p)$ is the time to attain an amount ξ_f at a constant parameter p. According to Eq. (A.1), the total time required to obtain an amount ξ_f is obtained by adding the absolute durations of time $\Delta\tau$ spent at different $\xi(\tau, p)$, until the sum of the relative durations $\Delta\tau/\tau_{\xi_f}(p)$ becomes unity. If ξ denotes the fraction transformed and the parameter p is temperature, Eq. (A.1) is the Scheil additivity rule [17,18] used to predict the transformation kinetics of non-isothermal phase transformations.

It has been established [18] that the factorisation of the reaction rate, to a function of the current amount of transformed product ξ and a function of the temperature p, is a sufficient condition for compatibility of the Scheil additivity rule (A.1) with the law $\xi(\tau, p)$ describing the transformation kinetics at a constant temperature p:

$$\frac{d\xi(\tau)}{d\tau} = \varphi_1(\xi)\varphi_2(p). \qquad (A.2)$$

In other words, if the factorisation (A.2) is possible then the Scheil additivity rule (A.1) holds. The fact that the factorisation (A.2) is also necessary has been proved in [19], i.e. if the Scheil additivity rule holds then the factorisation (A.2) necessarily follows. In other words if the factorisation (A.2) is impossible, the Scheil additivity rule (A.1) does not hold. In short, the Scheil additivity rule can be applied to calculate the transformed fraction during non-isothermal phase transformation, if and only if, the phase transformation rate can be factorised to a function of the amount of transformed fraction and a function of temperature [19].

Consequently, one of the reasons for the reported inaccuracies of predictions based on the Scheil additivity rule [20] is that the phase transformation rate cannot be factorised as in (A.2).

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